Group Theory in Rosen, *Symmetry in Science*

DEFINITIONS

Group: A *group* is a special type of set. In order to be classified as a group, a set must allow for any two of its elements to be combined (or *composed*) such that the conditions of *closure, associativity,* existence of *identity,* and existence of *inverses* are met (see below).

Order of a group*:* The *order* of a group (or set for that matter) is the number of elements within the group. This number of elements may be finite or infinite.

Four conditions for a group (CAII):

Closure: If the combination of any two set elements will always result in an element of the set, then the set has *closure*.

Associativity: If, when combining three or more elements of a set, the pairing of combinations does not change the result of the total combination, then the set has *associativity*. This condition does not imply that the order in which the elements are combined is immaterial, only that any given ordering can be evaluated via any pairing method. Thus: a(bc)=(ab)c, but a(bc) does not necessarily equal a(cb).

Identity: If a set contains an element that can combine in any order with any other element to reproduce the latter element, then the set has the existence of *identity*. Although this identity element is not required to be a unique element of the set, its uniqueness is easily proven.

Inverse: If for all members of a set there is some element of the set that can be combined with the original element to create the identity element, then the set has the existence of *inverses.* Elements can be their own inverse or rely on another element in the set for its inverse relationship, but each element has only one inverse.

As stated above, if all four of the above criteria hold true for a set, then the set is a group.

Commutative group: If the order in which two elements of a group are combined does not result in a different output, then the elements *commute*. Obviously, every element of a group commutes with itself, its inverse, and the identify element by definition. Groups in which all elements commute with every other element in the group are called *commutative groups*. This commutative property exists for a group if and only if the combinations in a group table are symmetrically reflective across the diagonal of the group table.

Group table: A *group table* graphically shows the *structure* of a group, i.e. it shows the results of all possible combinations of pairs of elements. This visualization of the structure of the group is done through listing all elements on a top row and a side row, then filling up the resulting matrix with the outputs from element-pair combinations. Obviously, group tables are only used with groups whose members form a finite set of elements.

Groups C1, C2, C3, C4, C5, C6 (C=cyclic): Each of these finite-order groups includes the identity element plus *n-1* elements, with *n* being the order of the group. In other words, C1 only includes the identity element whereas C5 includes the identity element plus four other elements.
Cyclic sets are those that represent incrementally-increasing roots of the number 1. The nomenclature is slightly confusing for music since a 3-cycle is represented by the C4 group while the 4-cycle is represented by the C3 group; this difference stems from group suffixes referring to the number of elements, not musical intervals. Cyclic groups seem to be able to be represented in two dimensions.

Groups D2, D3 (D=Dihedral): These dihedral groups are similar to the cyclic groups, except that their realizations derive from three-dimensional objects, as dihedral angles are angles between planes not lines.

Subgroup: Any set is a subgroup of a group if and only if all of the set's elements are elements of the group and the set itself forms a group. All groups have two obvious (*trivial*) subgroups, which are a subgroup that includes all the elements of the original group itself and the subgroup that includes only the identity element of the original group.

Mapping: When each element of one set is related via some specific method to individual elements of another set, the process is called *mapping*. The elements in the original set are called *objects* and the elements in the set to which the mapping leads are called *images*. Mapping is a one-way relationship between an initial set and another target set.

One-to-many mapping: Undefined by Rosen (since I believe it would fail the test as a real mathematical function), I would assume that *one-to-many mapping* is defined by the possibility of an element in the initial set having multiple images in the target set to which it is mapped. I do not think a one-to-many mapping would be a valid mathematical procedure, collaborated by the fact that many-to-one mappings are unable to have inverse mappings.

Many-to-one mapping: When elements of a target set may be mapped from one or more elements of the initial set, then this situation is termed *many-to-one mapping*.

One-to-one mapping: When each and every element of the first set corresponds with a unique and different element of the second set, then the first set maps into the second via a *one-to-one mapping*, also known as *injective* mapping. There may be more elements in the second set than the first, since the only requirement is that all the elements of the first set have partners in the second, not vice versa.

Mapping into: If the second set into which the first is mapped contains elements to which no element of the first set corresponds, then the first set is *mapping into* the second set.

Mapping onto: The reverse process of mapping into, *mapping onto* describes cases where every element of the target set is the image of at least one object in the initial set. This situation is also known as *surjective* mapping.

Inverse mapping: If a mapping is one-to-one and onto, i.e. each element in the first set maps to a unique element in the second set and each element in the second set is an image of a unique object in the first set, then an inverse mapping relationship can exist between the two sets. In other words, the mapping between the sets must be both injective and surjective, otherwise known as *bijective*. *Inverse mapping* describes the process where the target set is mapped back into the initial set, with objects becoming images and vice versa.

Homomorphism: Given a mapping between two groups, if the mapped image of a combination of two objects is the same as the combination of the mapped images themselves, then the mapping is called a *homomorphism*. In other words, the relationships between objects and their combinations in the initial group stay the same in the target group.

Isomorphism: If a homomorphism occurs in a one-to-one mapping (such that the mapping can be inverted), then an *isomorphism* exists. In other words, the elements involved in and the result of combinations conducted in either the initial group or in the target group will retain the same combinatorial properties when mapped to the other group.

Equivalence relation: In defining an *equivalence relation*, the necessary requirements for equality are mathematically codified. These necessary requirements are threefold:

1) For a relation to be equivalent, it must be *reflexive*, i.e. every element has to have the relation with itself (a=a).

2) For a relation to be equivalent, it must have *symmetry*, i.e. if an element has a relation with a second element, then the second element will also have the same relation with the original element (a=b and b=a).

3) For a relation to be equivalent, it must be *transitive*, i.e. if one element has a relation with a second element, and the second element has the relation with the third, then the first element has the relation with the third (a=b, b=c, and a=c)

PROBLEMS

Find the D2 group table, and explain the table in terms of the d, b, p, q set of letters and the group of operations: rotation 0, rotation 180, VF and HF:

The D2 group table is as follows:

e	*a*	*b*	*c*
a	*e*	*c*	*b*
b	*c*	*e*	*a*
c	*b*	*a*	*e*

If we let:

e = rotation by 0º (identity)
a = rotation by 180º
b = vertical flip
c = horizontal flip

....then we can confirm that:

(aa)=b since two rotations by 180º equals rotation by 360º or 0º

(ab)=c since *d* rotated 180º equals *p* and *p* flipped vertically equals *q*; similarly, *d* flipped horizontally equals *q*.

(ac)=b since *d* rotated 180º equals *p* and *p* flipped horizontally equals *b*; similarly, *d* flipped vertically equals *b*.

etc.....

Create a D4 group table:

e	*a*	*b*	*c*	*d*	*f*	*g*	*h*
a	*b*	*c*	*e*	*f*	*g*	*h*	*d*
b	*c*	*e*	*a*	*g*	*h*	*d*	*f*
c	*e*	*a*	*b*	*h*	*d*	*f*	*g*
d	*h*	*g*	*f*	*e*	*c*	*b*	*a*
f	*d*	*h*	*g*	*a*	*e*	*c*	*b*
g	*f*	*d*	*h*	*b*	*a*	*e*	*c*
h	*g*	*f*	*d*	*c*	*b*	*a*	*e*