Cycles and Geometrical Symmetry

The Western equal-tempered chromatic scale is based on the fundamental principle of dividing the octave evenly into twelve equal steps. Since each octave is twice the frequency of the last, this equal division requires a base-2 logarithmic scale, which allows each note to retain octave equivalency to all frequencies related by multiples of two. With twelve equal steps in an octave, therefore, each step in the musical scale must be $2^{1/12}$ times the frequency of the last step.

For convenience of illustration, musicologists often show the twelve notes of a chromatic scale in a circular pattern, much like Fig. 1. Sometimes nicknamed "the clock", this circular representation of the twelve notes often appears in books on atonal theory. While not useful for seeing overtone (i.e. consonance and dissonance) relationships between notes since such relationships are based on linear ratios between frequencies, the logarithmically-derived circular pattern is useful for viewing the types of "symmetrical" relationships used by twentieth-century composers.

In mathematics, a circular space is often graphed with what are known as "polar coordinates." In Fig. 2, a standard polar plot is shown with angles in terms of degrees at 30º intervals. As can easily be seen by comparing Figs. 1 and 2, the circular representation of the equal-tempered chromatic scale and a polar plot with increments of 30º share much in common. We can thus think about chromatic pitch-space as a polar space, using mathematical functions and terminology to discuss atonal musical techniques such as transposition and inversion.

Figs. 3-18 show all the cases for each cycle where pitch-class content remains the same, i.e. where pitch-class content is invariant. For example, Figs. 6-9 are various mappings of 3-cycles (called tricycles by me because it's more fun that way). Fig. 6 represents pitch-class set [048] mapped into polar space, the set inscribing itself as an equilateral triangle when the musical-note coordinates are connected with straight lines. Fig. 6 also represents transpositions of pitch-class set [048] at T_0, T_4, and T_8. The graphical equivalence of these transpositional levels can be proven if one manually transposes [048] through these transpositional levels and then compares the results to the original graph.

One can also transpose these sets via graphical methods, using the polar plots to more visually (and perhaps more easily) see transpositional levels. Each transpositional increment equates to an increase of 30º for the shape inscribed within the circle when its pitch-class members are mapped. For example, a transpositional level of 5 (T_5) equals (5 * 30º), or 150º of rotation. Thus, if we take our original [048] set class and submit its resultant triangle to 150º rotation within the polar plot, we arrive at the shape shown in Fig. 7, i.e. pitch-class set [159]. Doing a quick manual transposition easily confirms this process.

In addition to transposition, the graphical representations of the pitch-class sets also allow us to easily see the process of inversion. Inversion is graphically represented by reflecting each shape across the 0º-180º axis. Referring back to Fig. 7's graph of [159], we see that reflecting across the 0º-180º axis gives us the shape in Fig. 9, a graph of [37E]. Thus, [37E] is the inversion of [159]. Inversion can also be accomplished on these polar plots through more mathematical methods. This reflection procedure is achieved by simply multiplying each angle (or polar coordinate) by -1, e.g. "taking the negative" of each coordinate. Again, using Fig. 7 as our starting set, the polar coordinates of each point are [30º, 150º, -90º]. By multiplying each member by -1, we end up with [-30º, -150º, 90], coordinates that match exactly with those of Fig. 9.

It should be pointed out that some shapes are symmetrical around the 0º-180º axis while others are not, and this makes a big difference as to order of transposition and inversion. If we look at Figs. 6, 8, 10, etc., we see that inversion does not change the pitch-class content, i.e. the contents of these pitch-class sets are invariant under inversion. Invariance does not occur, however, with those geometric figures that are not symmetrical around the 0º-180º axis, such as Figs. 7, 9, 11, etc. With this latter category, inversion causes a change in the pitch-class content of the set. Therefore, the order of transposition and inversion matters greatly, for if one transposes into an invariant set, inversion afterwards will cause no change in pitch-class content. To give a situation that explicates such order dependence, let us take [048] and submit it to T_1 and then I. We can easily see that we are left with set [37E]. However, if we take [048] and submit it first to I and then T_1, we can see how we end up with set [159]. The reason for this difference is that the initial inversion on [048] made no change to the contents of the set. Of course, this order-dependent nature of transposition and inversion does not make a difference in every situation, for if one transposes both to and from an inversionally-invariant set, the order in which the inversion process occurs does not matter.

Symmetrical relations within the English alphabet:

Reflects into itself across the Y-axis:
A, H, I, M, O, T, U, V, W, X, Y
i, l, m, n, o, t, u, v, w, x

Reflects into itself across the X-axis:
B, C, D, E, H, I, O, X
c, l, o, x

Rotates into itself:
H, I, N, O, S, X, Z
l, o, s, x, z

Rotationally-related pairs:
M & W
a & e, b & q, d & p, m & w, n & u

Reflexively-related pairs:
b & d, p & q

Pairs related by rotation and reflection:
b & p, d & q

GEOMETRIC PROPERTIES OF INTERVAL CYCLES

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Fig. 1: Mapping PCs to Polar Space

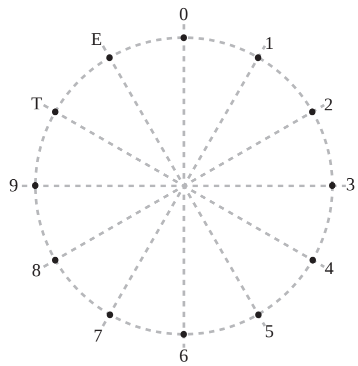

Fig. 2: Basic Polar Coordinates

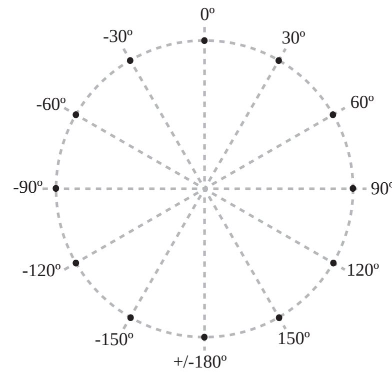

Fig. 3: Unicycle and Quinquecycle at all levels of T and I

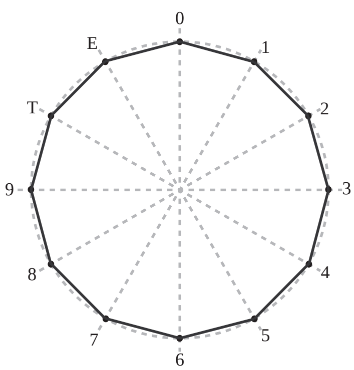

Fig. 4: Bicycle at even levels of T and TI

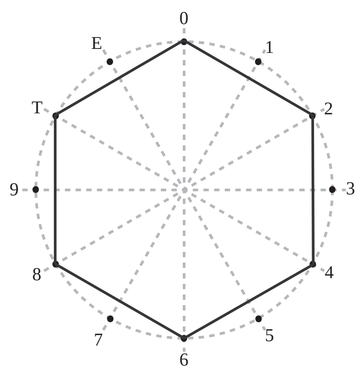

Fig. 5: Bicycle at odd levels of T and TI

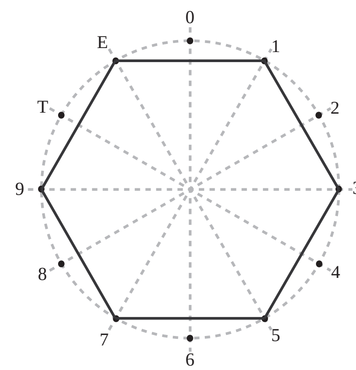

Fig. 6: Tricycle at T0, T4, T8, and T0I, T4I, T8I

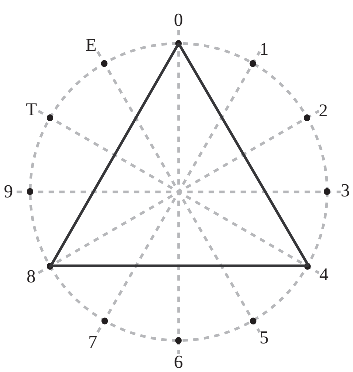

Fig. 7: Tricycle at T1, T5, T9, and T3I, T7I, T11I

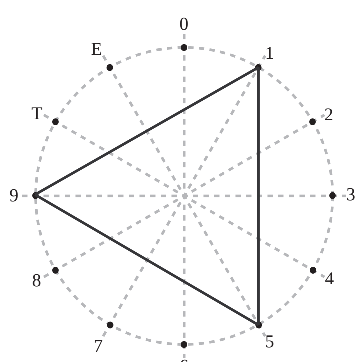

Fig. 8: Tricycle at T2, T6, T10, and T2I, T6I, T10I

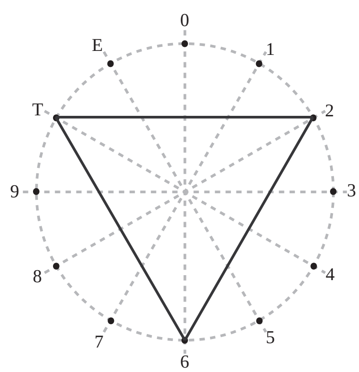

Fig 9: Tricycle at T3, T7, T11, and T1I, T5I, T9I

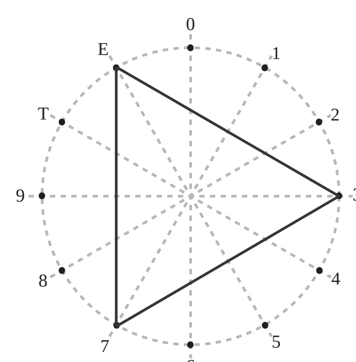

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Fig. 10: Quadricycle at T0, T3, T6, T9, and T0I, T3I, T6I, and T9I

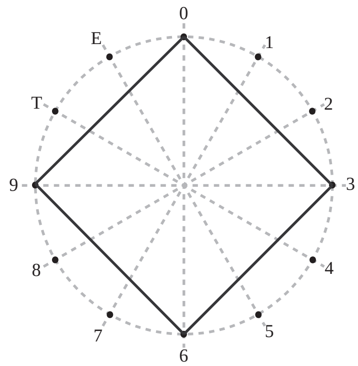

Fig. 11: Quadricycle at T1, T4, T7, T10, and T2I, T5I, T8I, T11I

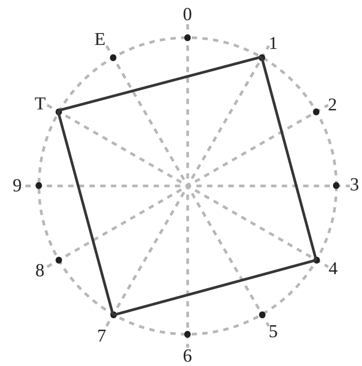

Fig. 12: Quadricycle at T2, T5, T8, T11, and T1I, T4I, T7I, T10I

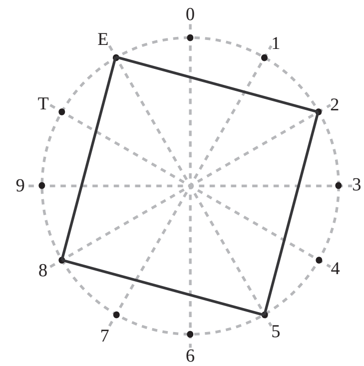

Fig. 13: Sexicycle at T0, T6, and T0I, T6I

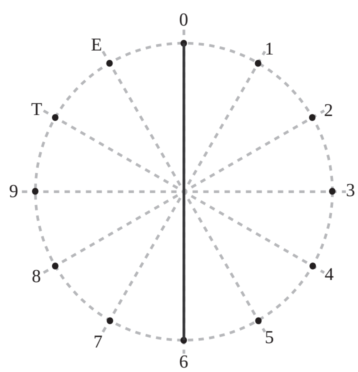

Fig. 14: Sexicycle at T1, T7, and T5I, T11I

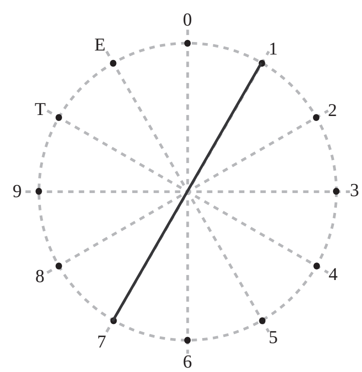

Fig. 15: Sexicycle at T2, T8, and T4I, T10I

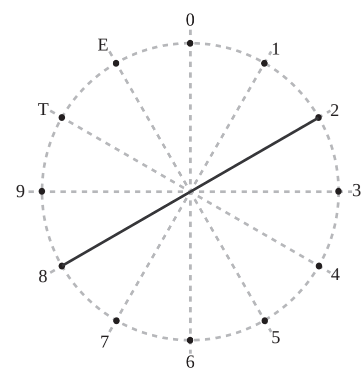

Fig. 16: Sexicycle at T3, T9, and T3I, T9I

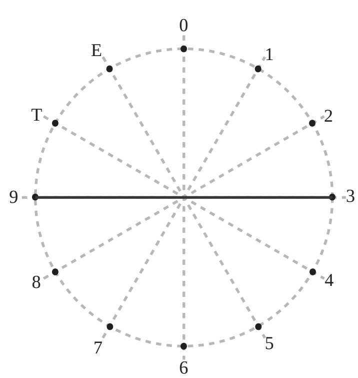

Fig. 17: Sexicycle at T4, T10, and T2I, T8I

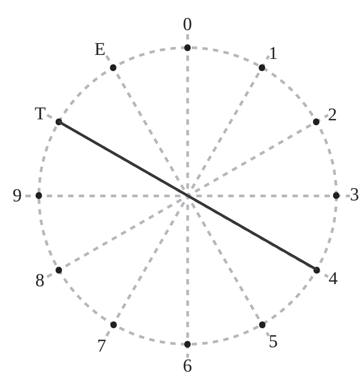

Fig. 18: Sexicycle at T5, T11, and T1I, T7I